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THE MONOTONICITY PROPERTIES OF SOME AXISYMMETRIC SUBSONIC FLOWS†

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A number of results are obtained for axisymmetric potential flows, characterizing the change in the gas-dynamic parameters which occurs in the subsonic region of a Laval nozzle with a cylindrical generatrix, and also in the case of flow around semi-infinite bodies with a cylindrical generatrix. This paper follows a previous paper [1] in which the monotonicity properties of the solutions of the systems of equations describing axisymmetrical flows were established, and also [2] in which a number of similar problems for plane flows were considered. © 1997 Elsevier Science Ltd. All rights reserved.

1. We will consider axisymmetrical potential flows of an ideal (inviscid and non-heat-conducting) gas. Using traditional dependent and independent variables, these flows are described by a non-homogeneous system of equations, which considerably complicates the analysis of axisymmetric subsonic flows. As a consequence, the results obtained for such flows are much less rigorous than for similar plane flows. Existing results were mainly obtained by analysing the solutions of approximate equations [3–7]. Hence, we would expect that, by changing to a homogeneous system obtained previously [1], additional possibilities would open up for investigating axisymmetric flows.

Consider [1] the functions $\alpha = u$ and $\beta = y\rho v$. Here and henceforth x and y are cylindrical coordinates (x is the axis of symmetry), u and v are the components of the velocity vector, ρ is the density, q and θ are the modulus and angle of inclination of the velocity vector, c is the velocity of sound and $M = q/c$ is the Mach number.

Using the functions α and β the flows in question can be described by the following homogeneous system of equations [1]

$$\begin{aligned}&\left(1-M^2\right)\alpha_x-\frac{uv}{y\rho c^2}\beta_x+\frac{c^2-v^2}{y\rho c^2}\beta_y=0\\&\frac{uv}{c^2}\alpha_x-\frac{c^2-v^2}{c^2}\alpha_y+\frac{\beta_x}{y\rho}=0\end{aligned}\tag{1.1}$$

The derivatives α_l and β_l, calculated along the lines β = const and α = const, respectively, have the form

$$\alpha_l=-\beta_n\frac{1-M^2\sin^2\omega}{y\rho},\quad \beta_l=\alpha_n y\rho\left(1-M^2\sin^2\varphi\right)\tag{1.2}$$

where β_n and α_n are the derivatives calculated along the normal to the lines β = const and α = const, and φ and ω are the angles which the tangents to the lines α = const and β = const make with the velocity vector, respectively.

For a unique determination of the level line of the function α, passing through the point considered, we will further assume that the level line α = const is also the limit of the region of increased or reduced values of α (with respect to the level line). In this case, when passing through possible branching points we choose those extreme branches which are adjacent to the region in question. In addition, which is important later, when moving along such a defined level line the sign of the derivative α_n does not change. This also applies to the level line of the function β.

It can be seen from (1.2) that when $M \ll 1$ the functions α and β possess monotonicity properties, according to which each of them is monotonic along the level line of the other function.

One of the important consequences of the monotonicity property is the lack of any closed level lines of the functions α and β provided that $M \ll 1$ on these level lines. Further, it follows from the fact that these level lines cannot terminate inside a subsonic region, that each of the level lines of the functions α and β, emerging from one point on the boundary of the subsonic region must necessarily reach this boundary at some other point. Finally, we point out the part played by the level line $\beta = 0$, along which not only the function $\alpha = u$ is monotonic, but also the pressure p and other functions which depend on q.

The above properties of the level lines of the functions α and β can be used to construct one of the possible versions of the level-line method for axisymmetric subsonic flows. In this case, the basis of the method is a combined analysis of the level lines of the functions α and β which pass through certain characteristic points, and the conditions

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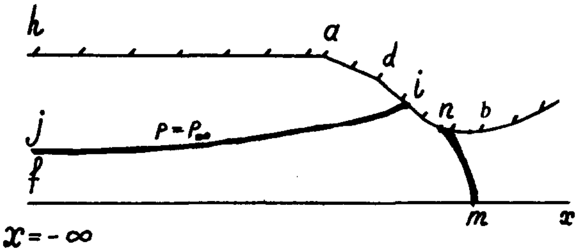

Fig. 1.

on the boundary of the subsonic region considered. When investigating certain problems this method enables one to prove the incompatibility of the results of an analysis of the level lines and the assumed properties of the flow and, as a consequence, to indicate the actual properties of the flow in question. This method, and the similar "modified" hodograph method, has been actively used when investigating plane flows [1, 2, 8–14], and also to study the simplified equations which describe near-sonic axisymmetric flows [7]. The level-line method is used in the following sections for axisymmetric subsonic flows described by the complete equations of gas dynamics.

2. Consider an axisymmetric Laval nozzle with a semi-infinite subsonic part (Fig. 1). Here in the section *fh* when $x = -\infty$, $v = 0$; *ha* is the cylindrical generatrix, *ab* is the constructed section, in which $v \leqslant 0$, $u \geqslant 0$, and to the right of the point *b* we have $v > 0$, $u > 0$. Along the section *ab*, with the exception of the point *b*, discontinuities of the contour are possible in which the flow is slowed down. In Fig. 1 we show two such points *a* and *d*. We will assume that potential flow occurs in the subsonic region *mfhan* without branching points of the streamlines and without local supersonic zones. The latter, in particular, means that on the upper all, to the left of the sonic line *n* (*mn* is the sonic line) there are no convex corner points.

The inequalities, characterizing the gas-dynamic parameters on the sonic line are of interest in themselves for the subsequent analysis of subsonic flow in the nozzle considered using the level-line method. In this connection the following theorem will be useful.

Theorem 1. On the sonic line *mn*, joining the axis of symmetry and the nozzle wall, $\beta \leqslant 0$, $v \leqslant 0$, and, as a consequence, the sonic point is situated on the constructed section *ab*.

Proof. The sonic point *m* on the axis of symmetry is a point of local maximum of *u* for the subsonic region and the sonic line. Consequently, in a fairly small subsonic neighbourhood of this point the lines $\alpha = u = \text{const}$ connect the axis of symmetry and the sonic line. For motion along these lines from the axis of symmetry $\alpha_n < 0$ and hence, by (1.2), on the sonic line in the region of the axis of symmetry $v < 0$. This fact is well known for the approximate equations [5–7].

For motion along the sonic line from the axis of symmetry, when the subsonic region remains on the left, a transition from negative values of v to positive values is possible on passing through the point at which either $v = 0, u = c$ or $v = 0, u = -c$. The first of these points is also a point of maximum of *u* for the subsonic region and the sonic line. Consequently, in a fairly small subsonic neighbourhood of this point, level lines $\alpha = u = \text{const}$ exist on which at the initial points $q = c$ and $v < 0$, at the final points $q = c$ and $v > 0$, and along which $\alpha_n < 0$. But, by (1.2), this is excluded. For the approximate equations this fact follows from other results obtained previously [7].

Thus, for a rigorous proof of the theorem it remains to consider the exotic possibility of a transition from $v < 0$ to $v > 0$ through the point at which $v = 0, u = -c$. To disprove this possibility, as indicated, we need to take into account the form of the subsonic part of the nozzle, since an analysis of the lines $\alpha = \text{const}$ in the neighbourhood of this point does not lead to any contradiction. Consider the level line $\beta = v = 0$ emerging from this point into the subsonic region. For motion along this level line $\beta_n \leqslant 0$ and, consequently, by (1.2) the function *u* increases

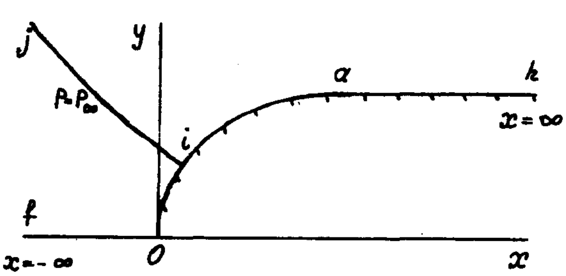

Fig. 2.

while q decreases. In the subsonic flow considered internal drag points are excluded, and hence on the level line up to where it reaches the boundary of the subsonic region $v = 0$, $\theta = \pi$, $-c < u = -q \leqslant 0$. But on the axis of symmetry, to the left at infinity and on the upper wall $\theta \neq \pi$. On the sonic line for $\theta = \pi$ we have $u = -c$. The contradiction obtained completes the proof of the theorem.

These results are used to prove the following theorem.

Theorem 2. At all points of the subsonic region of the nozzle considered (Fig. 1) $v \leqslant 0$, the velocity of the flow increases monotonically along the axis of symmetry *fm* and decreases monotonically along the cylindrical generatrix *ha*.

Proof. From the condition of the problem and from the results of Theorem 1 we have $v \leqslant 0$ on the boundary of the subsonic region. The suggestion that, at a certain internal point of this region $v > 0$, can be refuted by analysing the level line $\beta = y\rho v = \text{const}$ which passes through this point. This level line cannot be cut off inside the subsonic region, cannot be closed on itself and, finally, cannot reach the boundary of the region, since $v \leqslant 0$, $\beta \leqslant 0$ on it.

We will now assume that the velocity falls at a certain point on the axis of symmetry *fm*. Then along the level line $\alpha = u = \text{const}$, emerging from this point, by (1.2), the function β increases, and, consequently, this level line cannot reach the boundary of the region on which $\beta \leqslant 0$, which also disproves the above assumption. We can similarly demonstrate that it is impossible for the flow to accelerate at points of the cylindrical generatrix, which also completes the proof of the theorem.

Consequences of the theorem, in particular, are: the inequalities $p \geqslant p_\infty$ on *ha* and $p \leqslant p_\infty$ on *fm* and the existence in the subsonic region of at least one line *ij* of constant pressure (isobar) $p = p_\infty$, emerging from the constricted region at infinity. These facts are useful, for example, when interpreting experimental data for a Laval nozzle with a long subsonic cylindrical section.

3. We will consider the longitudinal flow around a semi-infinite body with a cylindrical generatrix *ah* (Fig. 2) and with a head section *oa*, along which the slope of the wall $0 \leqslant \gamma \leqslant \pi/2$. The flow is assumed to be potential, not containing closed streamlines and branch points of the streamlines, with the exception of the point *o*. At an infinite distance from the point *o* we have $u = u_\infty$, $0 < u_\infty < c$, $v = 0$. In the flow region $M \leqslant 1$. In view of the above assumptions on the axis of symmetry *fo* and on the cylindrical generatrix *ah* we have $v = 0$, $u \geqslant 0$, on the head section we will have $v \geqslant 0$, $u \geqslant 0$, and on the body the slope of the wall γ is identical with the angle θ.

Theorem 3. For such flow we have the following:

1. the velocity falls monotonically along the axis of symmetry *fo* and the cylindrical generatrix *ah*, and in the flow region there is at least one isobar *ij* $p = p_\infty$ which departs from the head section to infinity;

2. at all points of the flow region $0 \leqslant u \leqslant u_*$, $0 \leqslant \beta \leqslant \beta_*$, $0 \leqslant v \leqslant \beta_*/(\rho y)$, where u_* and β_* are the maximum values of u and β on the head section of the body (the maximum principle).

Proof. It follows from the results obtained previously in [15] that in the flow considered the quantity $|v|$ decreases as $R^{-(1+\varepsilon)}$, $R^2 = x^2 + y^2$, $\varepsilon > 0$ as one departs from the point *o* to infinity. Consequently, when $R = \infty$ we have $\beta = 0$. Suppose now that a certain point on *fo* or *ah* the derivative $u_x > 0$. Then along the level line of the function α, emerging from this point, by (1.2) the function β decreases and, consequently, this level line cannot reach the boundary of the region considered, including infinity, since $\beta \geqslant 0$ on this boundary. The contradiction obtained proves the first assertion of the theorem.

We will assume that at a certain internal point of the subsonic region one of the inequalities of the second assertion is not satisfied. Then a joint analysis of the corresponding level lines of the function α and/or β, passing through this point, and the boundary conditions lead to a contradiction. The theorem is proved.

These results can be transferred in a natural way to the case of a flow around semi-infinite stern sections containing a cylindrical generatrix *ah* (Fig. 2). In this case the flow is from right to left ($u_\infty < 0$), on the closing part *oa* we have $v \leqslant 0$, $u \leqslant 0$, and on the body, unlike the previous case, when the flow occurred from left to right at infinity, $\theta = \gamma + \pi$. Under these conditions $v \leqslant 0$ in the flow region and along the cylindrical generatrix and along the axis of symmetry the acceleration of the flow is positive. For plane flows, results similar to Theorems 2 and 3 were obtained previously in [2].

G. G. Chernyi and S. I. Chernyshenko, when discussing [1], pointed out the importance of using the results obtained in [1] to investigate axisymmetrical flows, and P. I. Plotnikov mentioned the result obtained in [15], which is fundamental for proving Theorem 3. I express my thanks to them.

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